Core 3 Module Revision Sheet

The C3 exam is 1 hour 30 minutes long and is in two sections.

Section A (36 marks) 5 – 7 short(ish) questions worth no more than 8 marks each.

Section B (36 marks) 2 questions worth approximately 18 marks each.

You are allowed a graphics calculator.

Before you go into the exam make sure you are fully aware of the contents of the formula booklet you receive. Also be sure not to panic; it is not uncommon to get stuck on a question (I've been there!). Just continue with what you can do and return at the end to the question(s) you have found hard. If you have time check all your work, especially the first question you attempted... always an area prone to error.

J M S

1. Proof

- There is not a lot for me to say other than read the chapter in the book. It's very short! I'll do a few examples here. I would make sure you know how to prove $\sqrt{2}$ is irrational and know how to extend it to $\sqrt{3}$ etc.

  1. *Prove that no square number ends in an 8.* Exhaustion: If a number ends with the digit one, then the square ends in one. $[2 \Rightarrow 4]$, $[3 \Rightarrow 9]$, $[4 \Rightarrow 6]$,..., $[9 \Rightarrow 1]$. None of them ended in eight, so we have proved the original assertion.
  2. *The value of $n^2 + n + 11$ is always prime.* Counter-example: When $n = 11$ the expression is not prime (divisible by 11).
  3. *The sum of five consecutive numbers is divisible by five.* Proof by direct argument: Let the first of the five numbers be n. Therefore the sum of the numbers is

     $$n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10.$$

     When we divide this by five we obtain $n + 2$. This is always an integer since n is an integer. Therefore we have proved the assertion.

2. Natural Logarithms & Exponentials

- Know that e is a special number in mathematics. It is approximately 2.7182818284... and it is irrational (i.e. it can't be expressed as a fraction; similarly to π).

- If the base of a logarithm is e then we call it a 'natural logarithm'. Written $\log_e x \equiv \ln x$.

- We already know that logarithms and exponentials are inverses of each other with the relationships

  $$\log_{10}(10^x) \equiv x \qquad \text{and} \qquad 10^{\log_{10} x} \equiv x.$$

  The same is true for natural logarithms and exponents of e;

  $$\ln(e^x) \equiv x \qquad \text{and} \qquad e^{\ln x} \equiv x.$$

- All the laws of logarithms from C2 are true for natural logarithms (e.g. $\ln ab = \ln a + \ln b$). For example make a the subject of the following equation (a few steps missed out):

$$\begin{aligned}
\ln(a-1) - \ln(a+1) &= b\\
\ln\left(\frac{a-1}{a+1}\right) &= b\\
\frac{a-1}{a+1} &= e^b\\
a(1-e^b) &= 1+e^b\\
a &= \frac{1+e^b}{1-e^b}.
\end{aligned}$$

- All of the techniques covered in C2 for modelling certain experiments can also be done with natural logarithms.

$$\begin{aligned}
y &= ae^{kx}\\
\ln y &= \ln(ae^{kx})\\
\ln y &= kx + \ln a.
\end{aligned}$$

So we plot $\ln y$ against x and we use the fact that the gradient is k and the y-axis intercept is $\ln a$. Practice these questions!

3. Functions

- A *function* is a *one-to-one* or a *many-to-one mapping*. There are also *many-to-many* and *one-to-many* mappings, but these are ***not*** functions.[1] For a function, for every value you feed into the function you obtain one (and only one) value out.
- The *domain* of a function $y = f(x)$ is all the possible values of x the function can take. For example the domain of $y = \sqrt{x-4}$ is $x \geqslant 4$. In other words all the *inputs* the function can take.
- The *co-domain* of a function is all the possible *outputs*. That is all the possible values of $f(x)$. So for $y = -x^2 + 5$ the co-domain is $y \leqslant 5$. The *range* of a function is the set of all outputs actually mapped to.
- Functions are transformed as follows

Function	Graph Shape
$f(x)$	Normal Graph
$2f(x)$	Graph stretched by a factor of 2 away from the x-axis i.e. every value of $f(x)$ in the original graph is multiplied by 2
$f(2x)$	Graph squeezed by factor of 2 towards the y-axis
$3f(4x)$	Graph squeezed by factor of 4 towards the y-axis followed by stretching by a factor of 3 away from the x-axis
$f(x)+6$	Graph moved vertically *up* 6 units
$f(x)-6$	Graph moved vertically *down* 6 units
$f(x+4)$	Graph moved 4 units to the *left*
$f(x-6)$	Graph moved 6 units to the *right*
$f(x-6)+9$	Graph translated 6 units to the *right* and 9 units *up*. This is a translation and can be expressed as $\binom{6}{9}$ where $\binom{\text{change in } x}{\text{change in } y}$
$-f(x)$	Graph reflected in the x-axis
$f(-x)$	Graph reflected in the y-axis

[1]See top of page 23 to see how to see what type of mapping a graph is; think about horizontal and vertical lines.

- When faced with compound transformations it sometimes matters which order you carry out the transformations. In the example of $2f(x-3)$ it doesn't matter because you end up with the same result both ways. However with $f(2x+10)$ you get a different result depending on the order you carry out the slide of 10 left and compression towards the y-axis. In this case you should do the *opposite* of what you expect. You do the slide first and then the compression.

- If $f(x) = f(-x)$ then the function is called an *even* function. An even function is one where the y-axis is a line of symmetry. Examples are

$$\begin{aligned} f(x) &= \cos x &\quad \text{since} \quad & f(-x) = \cos(-x) = \cos x = f(x), \\ g(x) &= x^2+1 &\quad \text{since} \quad & g(-x) = (-x)^2+1 = x^2+1 = g(x). \end{aligned}$$

- If $-f(x) = f(-x)$ then the function is called an *odd* function. An odd function is one where the function is unchanged if you rotate it $180°$ around the point $(0,0)$. Examples are

$$\begin{aligned} f(x) &= \sin x &\quad \text{since} \quad & f(-x) = \sin(-x) = -\sin x = -f(x), \\ g(x) &= x^3 &\quad \text{since} \quad & g(-x) = (-x)^3 = -x^3 = -g(x). \end{aligned}$$

- Note that $f(g(x))$ is not usually the same as $g(f(x))$. For example if $f(x) = x^2$ and $g(x) = x+1$ then $f(g(x)) = f(x+1) = (x+1)^2 = x^2+2x+1$. Contrast this with $g(f(x)) = g(x^2) = x^2+1$.

- Sometimes you will be asked to describe a quadratic of the form ax^2+bx+c in terms of $f(x) = x^2$. It is often useful to *complete the square*. Very quickly I will go through a couple of examples of how to do this:

$x^2+10 \Rightarrow$ Clearly just $f(x)+10$.
$x^2+6x+10 \Rightarrow$ Complete square to get $(x+3)^2-9+10 = (x+3)^2+1$ so it is $f(x+3)+1$, which is the translation $\binom{-3}{1}$ of x^2.
$2x^2+16x+1 \Rightarrow$ Complete square to get $2(x+4)^2-31$ so it is $2f(x+4)-31$, which is a stretch of factor 2 away from the x-axis, followed by a translation $\binom{-4}{-31}$ of x^2.

- To find the inverse of a function you swap round the x and the y and make y the subject again. This will be the inverse of the original function. Its graph will be the reflection of the original in the line $y = x$. For example find the inverse of $y = \sqrt{x^3+2}$ gives

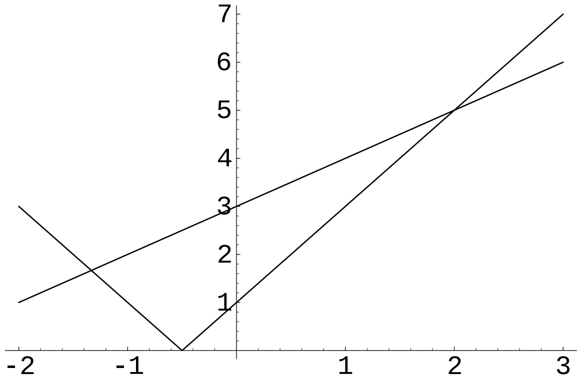
$$y = \sqrt{x^3+2} \qquad \Rightarrow \qquad x = \sqrt{y^3+2} \qquad \Rightarrow \qquad y = \sqrt[3]{x^2-2}.$$

- A function only has an inverse if it is a one-to-one mapping. If the original function is a many-to-one function (e.g. $y = x^2$ or any of the trig functions) you must restrict its domain to make it a one-to-one mapping (e.g. for $y = x^2$ restrict domain to $x \geqslant 0$). See example 3.9 on page 43 for an excellent exposition on how the domain and co-domain are related between a function and its inverse.

- The trig functions all have inverses if we restrict the domain. The conventional restrictions to allow inversion are

Function	Domain	Domain
$y = \sin x$	$-90° \leqslant x \leqslant 90°$	$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$
$y = \cos x$	$0° \leqslant x \leqslant 180°$	$0 \leqslant x \leqslant \pi$
$y = \tan x$	$-90° \leqslant x \leqslant 90°$	$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

- The modulus function you have met before. It makes everything you put into it positive. For example $|4| = 4$ and $|-6| = 6$. The best way of handling equations/inequalities involving the modulus function is to graph the left and right sides of the equation/inequality and find the intersects by way of solving the equations. For example solve $x+3 \geqslant |2x+1|$. First plot both to find



Then we solve $x+3 = 2x+1$ to find $x = 2$ and solve $x+3 = -2x-1$ to find $x = -\frac{4}{3}$. Putting this all together and by looking at the graph we find the overall solution is $-\frac{4}{3} \leqslant x \leqslant 2$.

4. Techniques For Differentiation

Chain Rule

- The chain rule is incredibly important! It states that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

This seems obvious from the way that differentials are written, but remember that they should not be thought of as fractions. It can be applied as follows to the example $y = (x^4 + x)^{10}$. Let $u = x^4 + x$, so

$$\begin{aligned} y &= u^{10} & \qquad u &= x^4 + x \\ \frac{dy}{du} &= 10u^9 & \qquad \frac{du}{dx} &= 4x^3 + 1. \end{aligned}$$

Therefore $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 10u^9 \times (4x^3 + 1) = 10(4x^3 + 1)(x^4 + x)^9$.

- The above method works all the time but it is a little slow. You will notice the general result that if $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$. So we can just write down the answer to similar problems. For example if $y = (3x^2 + 1)^5$ then $\frac{dy}{dx} = 30x(3x^2 + 1)^4$.

Product Rule

- Know that when $y = u \times v$ (where u and v are functions of x) we can differentiate it using the product rule. It states that

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

For example if $y = x^2(x^3-1)^3$ then

$$\begin{aligned}\frac{dy}{dx} &= [2x \times (x^3-1)^3] + [x^2 \times 3(x^3-1)^2 \times 3x^2] \\ &= 2x(x^3-1)^3 + 9x^4(x^3-1)^2 \\ &= x(x^3-1)^2[2(x^3-1)+9x^3] \\ &= x(x^3-1)^2(11x^3-2).\end{aligned}$$

Quotient Rule

- Very similar to the product rule is the quotient rule. It is used for functions of the form $y = \frac{u}{v}$. It states
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$
For example differentiating $y = \frac{x^3}{x^2+1}$ gives
$$\frac{dy}{dx} = \frac{(x^2+1)\times 3x^2 - x^3 \times 2x}{(x^2+1)^2}\frac{x^2(x^2+3)}{(x^2+1)^2}.$$

Trigonometric Functions

- When using radians we can differentiate the trigonometric functions. The results are as follows:
$$\begin{array}{ccc} y = \sin x & y = \cos x & y = \tan x \\ \dfrac{dy}{dx} = \cos x, & \dfrac{dy}{dx} = -\sin x, & \dfrac{dy}{dx} = \sec^2 x. \end{array}$$
One can derive the third result from the other two using the quotient rule and that $\tan x \equiv \frac{\sin x}{\cos x}$.
- You can also use these results along with the chain rule to differentiate functions like the following; $y = \sin(x^2+1)$ by letting $u = x^2+1$ and $y = (\tan x)^{10}$ by letting $u = \tan x$.
$$\begin{array}{cc} y = \sin(x^2+1) & y = (\tan x)^{10} \\ \dfrac{dy}{dx} = 2x\cos(x^2+1), & \dfrac{dy}{dx} = 10\sec^2 x(\tan x)^9. \end{array}$$

Implicit Differentiation

- Given a function in the form $y = f(x)$ we can differentiate it. Implicit differentiation allows us to differentiate a function without making y the subject first. It uses the chain rule that
$$\frac{d\,f(y)}{dx} = \frac{d\,f(y)}{dy} \times \frac{dy}{dx}.$$
So all you do is differentiate the y bits with respect to y and then multiply by $\frac{dy}{dx}$. For example differentiate $y^4 + x^4 = \sin y$ with respect to x. This gives
$$4y^3\frac{dy}{dx} + 4x^3 = \cos y\frac{dy}{dx} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{4x^3}{\cos y - 4y^3}.$$

- Another example; find all the stationary points on the curve $x^2 + y^2 + xy = 3$. Differentiating w.r.t. x we find

$$2x + 2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{2x+y}{2y+x}.$$

Stationary points are where $\frac{dy}{dx} = 0$ so solve

$$0 = -\frac{2x+y}{2y+x} \qquad \Rightarrow \qquad y = -2x.$$

Substituting this *back into the original equation* we find

$$x^2 + (-2x)^2 + x(-2x) = 3 \qquad \Rightarrow \qquad x = \pm 1 \qquad \Rightarrow \qquad \text{Points are } (1,-2) \text{ and } (-1,2).$$

5. Techniques For Integration

Integration by Substitution

- Integration by substitution is a way of integrating by replacing the variable given to you (usually x) and replacing it by another (usually u). These days the substitution you are to use is given to you in the exam, but practice will get you better at spotting what to substitute (usually the most complicated term in the integration or the denominator of a fraction). For example $\int x^3(x^4+1)^7\,dx$ we should use $u = x^4 + 1$.

$$\begin{aligned}
&\int x^3(x^4+1)^7\,dx && u = x^4+1\\
=&\int x^3u^7\,dx && \frac{du}{dx} = 4x^3\\
=&\int x^3u^7\,\frac{du}{4x^3} && \frac{du}{4x^3} = dx\\
=&\frac{1}{4}\int u^7\,du\\
=&\frac{u^8}{32} + c = \frac{(x^4+1)^8}{32} + c.
\end{aligned}$$

We have effectively *"used and abused"* u to help us to get the answer. (NOTE: I have been *very* sloppy in the above integration because I have mixed my x and u variables; you shouldn't really do this, but it makes the process of conversion clearer.)

- When dealing with definite integrals we need to also convert the limits of the integration and there is no need to convert back to x at the end since all definite integrals are merely numbers. For example

$$\begin{aligned}
&\int_3^4 2x\sqrt{x^2-4}\,dx && u = x^2-4 && x = 3 \quad \Rightarrow \quad u = 5\\
=&\int_5^{12} 2xu^{1/2}\,\frac{du}{2x} && \frac{du}{dx} = 2x && x = 4 \quad \Rightarrow \quad u = 12\\
=&\int_5^{12} u^{1/2}\,du && \frac{du}{2x} = dx\\
=&\left[\frac{2}{3}u^{3/2}\right]_5^{12}\\
=&\,20.3 \text{ (3sf)}.
\end{aligned}$$

Exponentials

- Know the result $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$.
- We know that if $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$. Therefore by reversal we find

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c.$$

For example[2]

$$\int x^3 e^{x^4}\,dx = \frac{1}{4}\int 4x^3 e^{x^4}\,dx = \frac{1}{4}e^{x^4} + c.$$

Integrals with Logarithms

- Know that $\int \frac{1}{x}\,dx = \ln x + c$.
- We know that if $y = \ln(f(x))$ then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$. Therefore by reversal we find

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.$$

For example[3]

$$\int \frac{x^3}{x^4+1}\,dx = \frac{1}{4}\int \frac{4x^3}{x^4+1}\,dx = \frac{1}{4}\ln|x^4+1| + c.$$

Integrals with Trigonometric Functions

- Know the results

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + c \qquad \text{and} \qquad \int \sin ax\,dx = -\frac{1}{a}\cos ax + c.$$

- Always be on the look out for integrals involving a mixture of trigonometric functions. These are usually handled by means of a substitution. For example $\int \cos x(\sin x)^7\,dx$ is best handled by $u = \sin x$ to give $\frac{1}{8}(\sin x)^8 + c$.
- Also know the useful results (all derived from reverse chain rule)

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c \qquad \text{and} \qquad \int f'(x)\sin f(x)\,dx = -\cos f(x) + c.$$

For example $\int x^3\cos(x^4)\,dx = \frac{1}{4}\sin(x^4) + c$.

Integration by Parts

- When an integral is made up of two 'bits' then we can sometimes use integration by parts. It states

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

[2] This could also have been evaluated (more slowly) by a substitution of $u = x^4$ which would then have reduced to $\int x^3 e^{x^4}\,dx = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^{x^4} + c$.

[3] Again, this could also have been evaluated by the substitution $u = x^4 + 1$.

So you will need to decide which 'bit' of the integral you will need to differentiate and which 'bit' to integrate. For example in $\int x \sin x \, dx$ it is quite clear that we will need to differentiate the x 'bit' and integrate the $\sin x$ 'bit'.

$$\begin{aligned}\int [x][\sin x] \, dx &= [x][-\cos x] - \int [1][-\cos x] \, dx \\ &= -x\cos x + \sin x + c.\end{aligned}$$

(Don't use the square brackets when you do it; I only used it to show where everything comes from.)

- Another example (this time a definite integral)

$$\begin{aligned}\int_0^2 xe^{2x} \, dx &= \left[\frac{1}{2}xe^{2x}\right]_0^2 - \int_0^2 \frac{1}{2}e^{2x} \, dx \\ &= \left[\frac{1}{2}xe^{2x}\right]_0^2 - \left[\frac{1}{4}e^{2x}\right]_0^2 \\ &= \left(e^4 - 0\right) - \left(\frac{e^4}{4} - \frac{1}{4}\right) = \frac{3e^4}{4} + \frac{1}{4}.\end{aligned}$$